Mean-field crack networks on desiccated films and their applications: Girl with a Pearl Earring

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Usual requirements for bulk and fissure energies are considered in obtaining the interdependence among external stress, thickness and area of crack polygons in desiccated films. The average area of crack polygons increases with thickness as a power-law of 4/3. The sequential fragmentation process is characterized by a topological factor related to a scaling finite procedure. Non-sequential overly tensioned (prompt) fragmentation is briefly discussed. Vermeer’s painting, Girl with a Pearl Earring, is considered explicitly by using computational image tools and simple experiments and applying the proposed theoretical analysis. In particular, concerning the source of lightened effects on the girl’s face, the left/right thickness layer ratio (\( E_{1.34} \)) and the stress ratio (\( E_{1.102} \)) are evaluated. Other master paintings are briefly considered.

1 Introduction: crack networks

It is normal to observe crack networks in desiccated media. Crack networks develop in mud\(^1\) and clay,\(^2\) dry agricultural fields, master paintings,\(^3,4\) geological rocks,\(^5-7\) salt fields, contracted permafrost,\(^8,9\) micrometric systems,\(^10\) nanometric structures\(^11\) and cosmetic products, among others. They include corrugated substrates\(^12,13\) and other types of crack geometries.\(^14,15\) On a fine scale, every crack network is topologically singular and useful, for instance, in recognizing a master painting.\(^4\) Nevertheless, at a coarse scale, such networks have generic features and it thus seems suitable to develop a type of mean-field theory\(^16\) which is the purpose of this study.

Almost all crack networks in fractured films on substrates have the following features.
- Polygons, or cells, produced by cracking have no genuine fractal nature. Indeed, a characteristic distance (thickness \( h \)) exists and breaking self-similarity is one of the chief properties of fractals (ref. 17 and 18).
- They have an average area \( A \) (per cell) presumably related to the thickness of the film.
- Vertices are usually three-connected.
- Typically, fissures advance and form large polygons first and smaller ones later. This local fragmentation process is coarsely sequential. Additionally, the rupture separation or width \( d \) is larger for earlier than for subsequent cracks.
- An “impurity”, another crack for instance, is presumably necessary but not sufficient for crack formation. Experiments have revealed that a sharp material boundary can also play the role of an impurity.
- Fragmentation does not diminish if desiccation is not reversed, suggesting an analogy with entropy concepts.

1.1 Elasticity elements and the mean-field approach

This paper assumes that elasticity theory is valid as desiccation progresses. In this sense,\(^19\) cracks of lateral area \( S \) are powered by bulk stress energy. Explicitly, let us consider a non-fractured cell of volume \( V \) (Fig. 1) subjected to direction-averaged stress \( o \) (i.e. \( o^2 \propto (1/9) \sum_i o_{ij}^2 \)).

![Fig. 1 Scheme of a cracked film on a substrate. For simplicity, cells are represented as cubes. The characteristic area of each surface is \( A \) and the thickness of the film is \( h \). A full crack, in a cell, generates two surfaces of area \( S = \sqrt{Ah} \). The external boundary stress value is \( P \), the volume of a cell is \( V = Ah \) and the rupture separation is \( d \).](image-url)
As usual, crack formation\textsuperscript{19} requires the condition between bulk energy ($\propto V$) and fissure energy ($\propto S$):

$$\frac{1}{2}x\sigma^2 V \geq \gamma S,$$

where $x$ is a parameter proportional to the modulus of rigidity and dependent on the evaporation rate and heat flux. The parameter $\gamma$ is the energy on the unit surface required to realize the fissure of lateral area $S$ (i.e., molecular bond ruptures).

For films on substrates, it is better to express the stress as a function of the boundary conditions. Following Landau,\textsuperscript{19} in equilibrium and using the divergence theorem, the volume-averaged stress tensor can be expressed as a function of the external boundary forces $P(x)$ acting on the unit area:

$$\langle o_{ij} \rangle = \frac{1}{2V} \int (x_{ij} P_{ij} + P_{ij} x_{ij}) dA,$$

where the surface integral is evaluated on the boundaries of the volume $V$ and the left brackets denote volume average. According to eqn (2) and for films on substrates, the main external tension comes from the surface of contact\textsuperscript{20} and consequently $\langle o_{ij} \rangle \propto 1/h$. A rough estimation of the above integral gives

$$\langle o^2 \rangle \approx P^2 A \frac{h}{F_n},$$

where $P^2$ is the characteristic quadratic value of the external stress, for instance, related to shrinkage by desiccation.

Eqn (3) gives an estimation of the stress in the medium. Note that the quantities $h$ and $P$ can be viewed as experimental control parameters. In fact, they are assumed to be independent parameters.

2 Estimation of the area of film cracking

The energy balance eqn (1) is combined with the mean-field eqn (3). Thus, in a non-ruptured cell having an area $A$ and volume $V = Ah$, the condition that must be met for a full crack of area $S = \sqrt{Ah}$ is

$$2P^2 A^{3/2} \geq 2\gamma h^2,$$

where the equality $2P^2 A^{3/2} = 2\gamma h^2$ corresponds to the critical situation. This equation provides, for instance, an estimation of the external stress needed to break the cell, and it defines a critical surface in the space ($P, h, A$). In the case of a painting, the equality defines the crack area as a function of the film thickness related ultimately to color intensity.

Inequality (4) is the main mathematical relation used in this study. The formation of crack cells is favored by large external stress $P$, small thickness $h$, large area $A$ and a large modulus of rigidity or weak molecular bonding.

When the external stress $P$ is not correlated with the area $A$, assumed for paintings, the characteristic polygon size $\sqrt{A}$ is proportional to $h^{2/3}$. Additionally, similar to the case of fractures in a solid as a result of external mechanical loads, cracks do not develop for fully porous and inhomogeneous films with an average impurity size violating eqn (4) (Griffith’s criterion\textsuperscript{21,22}). This seems the case, for instance, for the classical painting Self Portrait by Rembrandt (Mauritshuis Museum, oil on canvas, 1669), where no cracks are evident on Rembrandt’s face.

Finally, if an external force $F$ is proportional to $(\sqrt{A})^{3/2}$ then the external stress $P \propto 1/A^{1/4}$ and the cell size from (4) becomes proportional to thickness $(\sqrt{A} \propto h)$ as in ref. 4.

3 A finite scaling perspective for fragmentation: sequential versus prompt

Cell partitions are related to an iteration based on inequality (4). When desiccation increases slowly (i.e., the case of sequential fragmentation), the dimensionless tension parameter $P = \sqrt{\gamma h} / 2\gamma P$ continuously increases. This continuous growth generates successive partitions of cells. Fig. 2 shows four stages ($A_1 \rightarrow A_2 \rightarrow A_3 \rightarrow A_4$) of a hypothetical cell partition in accordance with eqn (4). When a cell in stage $n$ cracks, the critical parameters scale as (sub-index $i$ is dropped):

$$A_n \rightarrow A_{n+1} = A_n / g$$
$$P_n \rightarrow P_{n+1} = P_n g^{3/4}$$
$$S_n \rightarrow S_{n+1} = S_n / \sqrt{g}$$
$$a_n \rightarrow a_{n+1} = a_n g^{1/4}$$
$$h_n \rightarrow h_{n+1} = h_n g^0.$$

Here the quantity $g$ ($>1$) is an effective topological factor that depends on the evaporation rate, specific material and other factors. The latter scaling relationship was put explicitly to remember that the thickness is assumed to be constant, and

Fig. 2. Area fragmentation in 4-steps $A_1 \rightarrow A_2 \rightarrow A_3 \rightarrow A_4$ (green lines) as a function of the dimensionless tension parameter $P = \sqrt{\gamma h} / 2\gamma P$. The descending curve with black points is the critical curve corresponding to the equality in eqn (4). The region under the curve is the stable region in which cracks do not form. Above the curve, the red dot indicates non-sequenced fragmentation $A_1 \rightarrow A_4$, possibly producing a global symmetric final network.
the scaling for the stress $o$ derives from eqn (3). Moreover, after $n$ iterations, the number of cells or polygons $N$ is approximately $g^n$.

Finally, note that the scaling procedure (5) is related to films; consequently, the condition $A_o \gg h^2$ must be imposed and therefore a finite number of steps obtains.

For non-isolated cracked films, the most common type, fragmentation can still occur after desiccation stops. In fact, condition (4) is valid for any type of external tension $P$ acting from the substrate, including mechanical fluctuations. For instance, in old master paintings, mechanical manipulations of the wooden frame perturb the canvas and fractures eventually propagate.\textsuperscript{3,4} Additionally, we have conducted experiments in which we apply intentional strokes to finish crack cells that are not closed.

Meanwhile, let us consider a non-ruptured area ($A_1$) subjected to high external tension (see the red point in Fig. 2). In this case, any external perturbation must quickly activate a direct fragmentation ($A_1 \to A_4$) in a non-sequenced process simultaneously throughout the film. As a conjecture, the final polygon network is expected to emerge quite symmetrically if there are no traces of impurities and the materials are homogeneous in this extremely tensioned global process.

### 4 Girl with a Pearl Earring (Girl with a Turban)

Consider the optically lightened effects on Girl with a Pearl Earring by J. Vermeer (~1665, viewable on the Internet, Mauritshuis Museum; Fig. 3(a)), where the right side of the girl's face seems more illuminated than the left. Importantly and in general, the girl's face has larger crack polygons on the right than on the left (Fig. 3b).

The file-recognition “Image-J” was applied to the image. Nevertheless, this procedure becomes quite sensible to amplifications and color inhomogeneity of a given location. In this study, desiccation is assumed as the main stress source in the masterpiece.

#### 4.1 Face area ratio estimations: thickness and stress ratios

By applying directly the file-recognition procedure, an $A_{\text{right}}/A_{\text{left}}$ cell area ratio of $\approx 1.45$ was found under the girl's eyes and an $A_{\text{right}}/A_{\text{left}}$ Ratio $\approx 1.5$ on the right eye and the nose (forefront). Using these values and our prescription related to eqn (4) (explicitly $\sqrt{A} \propto h^{2/3}$), the ratio of film thicknesses is estimated for this masterpiece as $h_{\text{right}}/h_{\text{left}} \approx 1.34 \pm 0.02$.

The critical stress ratio between both independent cracked zones can similarly be calculated. From (3) and (4), $o_{\text{left}}/o_{\text{right}} \approx 1.1020 \pm 0.005$ for this work of art.

#### 4.2 Additional support

Always related to Vermeer's work, to support the above area calculations, consider Fig. 4a and b.

Both figures show the color structure of the girl's face along two vertical lines under the eyes (see Fig. 3a). The Y-axis gives the color value of the pixel, with a greater value approaching white and a smaller value approaching dark colors; i.e., tones associated with cracks. The X-axis gives the number of pixels on the vertical line. Fig. 4a gives color values for the right line and shows a greater separation between picks (cracks) than does Fig. 4b for the left line. Counting picks, around color values of 190–200, the ratio becomes $9/6 = 1.5$ and similar to the mentioned ratio between areas (subsection 4.1). Interestingly, a similar ratio under the eyes (~650 pixels) can be found in the renaissance masterpiece Portrait of a Young Girl by Petrus Christus.

#### 4.3 A simple test for lightened effects

An experimental test confirms the optically lightened effects on the Girl with a Pearl Earring due to an additional layer on the right side of the girl's face (subsection 4.1).

Fig. 5 shows two experimental yellow spots on a substrate (inset photographs).
The right film, with respect to the left film, contains two paint layers. Upon comparing the polygon networks, three measures of the $A_1/A_2$ cell ratio carried out with Image-J software give $1.49 \pm 0.21$, similar to the considered masterpiece. Additionally, the main graph shows the crack distribution along a line crossing both material films (800 pixels, line not shown). In the figure, the crack ratio, around a color value of 150, becomes 9/6 in reasonable accordance with the finding for *Girl with a Pearl Earring*. The central box in the graph corresponds to the separation between the two films.

In sum, from subsections 4.1, 4.2 and 4.3, the optically lightened effects on the face of the girl were realized (mostly) by adding more painting material ($h_{\text{right}} > h_{\text{left}}$).

5 Brief consideration of polygon entropy

For constant external stress, out of the critical region, the average number of cracks becomes fixed. Additionally, the volume $A_T h$ likewise remains constant, $A_T$ being the initial area. In particular, the average number of polygons becomes $N \approx A_T/A$, suggesting an analogy with thermodynamic micro-canonical entropy:

$$E(N) = \ln N.$$ 

As long as the stress energy is $U = (aP^2)/2)A_T h$ and neglecting the separation $d$ in the fissures, it follows from condition (4),

$$E \propto \frac{2}{3} \ln \frac{U}{\sqrt{h^3}}. \quad (6)$$

In this way and formally, equivalent thermodynamic quantities can be eventually defined.

6 Considerations beyond the scope of the present study

Several points on the approach based on mean-field eqn (4) have been excluded.

- The approach does not forecast the resulting types of polygons in the ruptured network. In the particular case of a painting, similar to the case for mud cracks, fissures are usually sequenced given the usual $T$-vertices rather than $Y$-vertices.
- The approach neglects the separation $d$ of fissures. Thus imposing $A_T \approx NA + Nd\sqrt{A}$ could lead to improvements.
- Highly symmetric networks must be related to overly tensioned processes (Section 3) but, to the best of our knowledge, no evidence of highly symmetric networks in paintings exists.

7 Conclusions

A mean-field prescription for the cracking of a desiccated film was derived (inequality (4)). It defines a connection between the external stress, film thickness, average polygon area and material properties. In the critical domain, the prescription allows the estimation of properties of classical paintings. In particular, in Vermeer’s painting *Girl with a Pearl Earring* on canvas, the
difference in lightened effects between the left and right sides of the girl’s face becomes mainly related to a thickness film ratio of $1.34 \pm 0.02$ and $1.1020 \pm 0.005$ for the stress, respectively.

Sequenced cracked structures, or networks, were considered using a topological factor and scaling prescription. This outlines the geometrical growth of the number of polygons. Additionally, the possibility of highly symmetric structures forming in an overly tensioned process was conjectured.

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References

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